UNCLASSIFIED

AD NUMBER

AD459518

LIMITATION CHANGES

TO:

Approved for public release; distribution is unlimited.

FROM:

**Distribution authorized to U.S. Gov't. agencies and their contractors;
Administrative/Operational Use; 02 MAR 1965.
Other requests shall be referred to Office of Naval Research, 875 North Randolph Street, Arlington, VA 22203-1995.**

AUTHORITY

ONR ltr, 15 Jun 11977

THIS PAGE IS UNCLASSIFIED

UNCLASSIFIED

AD 459518

DEFENSE DOCUMENTATION CENTER

FOR

SCIENTIFIC AND TECHNICAL INFORMATION

CAMERON STATION ALEXANDRIA, VIRGINIA

UNCLASSIFIED

NOTICE: When government or other drawings, specifications or other data are used for any purpose other than in connection with a definitely related government procurement operation, the U. S. Government thereby incurs no responsibility, nor any obligation whatsoever; and the fact that the Government may have formulated, furnished, or in any way supplied the said drawings, specifications, or other data is not to be regarded by implication or otherwise as in any manner licensing the holder or any other person or corporation, or conveying any rights or permission to manufacture, use or sell any patented invention that may in any way be related thereto.

CATALOGED BY: DDC
AS AD NO. 459518

Office of Naval Research
Contract Nonr-1866 (16) NR-372-012

A SUCCESSIVE SWEEP METHOD FOR SOLVING OPTIMAL PROGRAMMING PROBLEMS

DDC
RECEIVED
MAR 25 1965
DDC-IRA B

By
Stephen R. McReynolds and Arthur E. Bryson, Jr.

March 2, 1965

Technical Report No. 463

Cruft Laboratory
Division of Engineering and Applied Physics
Harvard University • Cambridge, Massachusetts

459518

Office of Naval Research

Contract Nonr-1866(16)

NR - 372 - 012

A SUCCESSIVE SWEEP METHOD FOR SOLVING OPTIMAL PROGRAMMING PROBLEMS

by

Stephen R. Mc Reynolds and Arthur E. Bryson, Jr.

March 2, 1965

The research reported in this document was made possible through support extended to Cruft Laboratory, Harvard University, by the U. S. Army Research Office, the U. S. Air Force Office of Scientific Research, and the U. S. Office of Naval Research under the Joint Services Electronics Program by Contract Nonr-1866(16). Reproduction in whole or in part is permitted for any purpose of the United States Government.

Technical Report No. 463

Cruft Laboratory

Division of Engineering and Applied Physics

Harvard University

Cambridge, Massachusetts

A SUCCESSIVE SWEEP METHOD FOR SOLVING OPTIMAL PROGRAMMING PROBLEMS

Stephen R. McReynolds and Arthur E. Bryson, Jr.

Division of Engineering and Applied Physics
Harvard University

ABSTRACT

An automatic, finite-step numerical procedure is described for finding exact solutions to non-linear optimal programming problems. The procedure representa a unification and extension of the steepest-descent, and aecond variation techniques.

The procedure requires the backward integration of the uaual adjoint-vector differential equations plus certain matrix differential equations. These integrations correspond, in the ordinary calculus, to finding the first and second derivatives of the performance index respectively. The matrix equations arise from an inhomogeneous Ricatti transformation, which generates a linear "feedback control law" that preserves the gradient histories, $H_u(t)$, on the next step or permits changing them by controlled amounts, while also changing terminal conditions by controlled amounts. Thus, in a finite number of steps, the gradient histories can be made identically zero, as required for optimality, and the terminal conditions satisfied exactly. One forward plus one backward sweep, correspond to one step in the Newton-Raphson technique for finding maxima and minima in the ordinary calculus.*

As by-products, the procedure produces (a) the functions needed to show that the program is, or is not, a local maximum (the generalized Jacobi test) and (b) the feedback gain programs for neighboring optimal paths to the same, or a slightly different, set of terminal conditions.

CLASS OF PROBLEMS TREATED

The method is applicable to a class of non-linear optimal programming problems where one wishes to determine control functions $u(t)$ in $t_0 \leq t \leq t_1$ so as to minimize (or maximize) a performance index of the form

$$J = \phi[x(t_1),t_1] + \int_{t_0}^{t_1} L[x(t),u(t),t]dt \qquad (1)$$

subject to the constraints

$$\dot{x} = f[x(t),u(t),t] \qquad (2)$$

$x(t)$ is an n-component state vector

$u(t)$ is an m-component control vector

$$x(t_0) = x_0 \qquad (3)$$

$$\psi[x(t_1),t_1] = 0 \quad ; \psi \text{ is a q-component vector } (q \leq n) \qquad (4)$$

A further restriction on the class of problems treated in this paper is that we assume $\frac{\partial^2 H}{\partial u^2}$ is a positive-definite (or negative-definite) matrix over the whole interval $t_0 \leq t \leq t_1$ where H is the variational Hamiltonian introduced in the next section.

The final time, t_1 , may be given either explicitly or implicitly in Eqns. (4). For simplicity of presentation, we will first discuas the case where the final time t_1 is given explicitly.

CASE WHERE FINAL TIME IS GIVEN EXPLICITLY

In the usual manner we introduce the auxiliary scalar functions

$$H(\lambda,x,u,t) = L(x,u,t) + \lambda^T f(x,u,t) \qquad (5)$$

$$\Phi(\nu,x,t) = \phi(x,t) + \nu^T \psi(x,t) \qquad (6)$$

where $\lambda(t)$ is an n-component vector of influence functions and ν is a q-component constant vector. We regard $u(t)$ as control functions and ν as control parameters, and introduce a modified performance index $\bar{J}$ where

$$\bar{J} = \Phi[\nu,x(t_1),t_1] + \int_{t_0}^{t_1} \{H[\lambda(t),x(t),u(t),t] - \lambda^T(t)\dot{x}\}dt \qquad (7)$$

Note that when (2) and (4) are satisfied, (7) is identical with (1).

Necessary conditions for an extremal path are (e.g. see Ref. 1)

$$\dot{\lambda}^T = -H_x \qquad (8)$$

$$0 = H_u \qquad (9)$$

$$\lambda^T(t_1) = \Phi_x[\nu,x(t_1),t_1] \equiv (\phi_x + \nu^T \psi_x)_{t=t_1} \qquad (10)$$

Suppose we arbitrarily choose some control functions $u(t)$ and some control parameters ν , integrate Eqns. (2) forward with initial conditions (3), and Eqns. (8) backward with boundary conditions (10). In general, Eqns. (4) and (9) will not be satisfied. Now, consider a perturbation around this path:

$$\delta\dot{x} = f_x \delta x + f_u \delta u \qquad (11)$$

$$\delta\dot{\lambda} = -H_{xx}\delta x - f_x^T \delta\lambda - H_{xu}\delta u \qquad (12)$$

$$\delta H_u = H_{ux}\delta x + H_{uu}\delta u + f_u^T \delta\lambda \quad \text{specified} \qquad (13)$$

$$\delta x(t_0) \quad \text{specified} \qquad (14)$$

$$\delta\lambda(t_1) = [\Phi_{xx}\delta x + \psi_x^T d\nu]_{t=t} \qquad (15)$$

$$\delta\psi = [\psi_x \delta x]_{t=t_1} \quad \text{specified} \qquad (16)$$

We may regard (11)-(16) as a linear,

---

* This latter technique is reviewed briefly in the Appendix to this paper.

inhomogeneous two-point boundary value problem that determines the functions $\delta x(t)$, $\delta\lambda(t)$, $\delta u(t)$, and the parameters $d\nu$ in terms of specified functions $\delta H_u(t)$ and specified parameters $\delta x(t_0)$ and $\delta\psi$. This is very close to the viewpoint taken by Merriam (Ref. 2) and Kelley, Kopp, and Moyer (Ref. 3).

To solve this two-point boundary value problem we may solve (13) for $\delta u(t)$ in terms of $\delta x(t)$, $\delta\lambda(t)$, and $\delta H_u(t)$, provided $H_{uu}(t)$ is non-singular:

$$\delta u = -H_{uu}^{-1}[-\delta H_u + H_{ux}\delta x + f_u^T\delta\lambda] \tag{17}$$

and, upon substituting (17) into (11) and (12), we obtain

$$\delta\dot{x} = A\,\delta x + B\,\delta\lambda + v \tag{18}$$

$$\delta\dot{\lambda} = C\,\delta x - A^T\delta\lambda + w \tag{19}$$

where

$$A = f_x - f_u H_{uu}^{-1} H_{ux} \tag{20}$$

$$B = -f_u H_{uu}^{-1} f_u^T \tag{21}$$

$$C = -H_{xx} + H_{xu} H_{uu}^{-1} H_{ux} \tag{22}$$

$$v = f_u H_{uu}^{-1}\delta H_u \tag{23}$$

$$w = -H_{xu} H_{uu}^{-1}\delta H_u \tag{24}$$

THE INHOMOGENEOUS RICATTI TRANSFORMATION

In view of Eqns. (15) and (16), let us introduce the following inhomogeneous Ricatti transformation (suggested in Refs. 4, 5):

$$\delta\lambda(t) = P(t)\delta x(t) + R(t)d\nu + h(t) \tag{25}$$

$$\delta\psi = R^T(t)\delta x(t) + Q(t)d\nu + g(t) \tag{26}$$

where $d\nu$ and $\delta\psi$ are constant infinitesimal vectors, $P(t)$, $R(t)$, and $Q(t)$ are matrix functions, and $h(t)$ and $g(t)$ are vector functions. Now, differentiate (17) and (18) with respect to time:

$$\delta\dot{\lambda} = P\,\delta\dot{x} + \dot{P}\,\delta x + \dot{R}\,d\nu + \dot{h} \tag{27}$$

$$0 = R^T\delta\dot{x} + \dot{R}^T\delta x + \dot{Q}\,d\nu + \dot{g} \tag{28}$$

Using (25) in (18) gives

$$\delta\dot{x} = (A+BP)\delta x + BR\,d\nu + Bh + v \tag{29}$$

Equating (19) and (27) and using (25) and (29) to eliminate $\delta\dot{x}$ and $\delta\lambda$, we have:

$$(C-A^TP-PA-PBP-\dot{P})\delta x - [(A^T+PB)R+\dot{R}]d\nu$$
$$- [(A^T+PB)h+Pv-w+\dot{h}] = 0 \tag{30}$$

In a similar fashion, substitute (29) into (28):

$$[\dot{R}^T+R^T(A+BP)]\delta x + (R^TBR+\dot{Q})d\nu + R^T(Bh+v) + \dot{g} = 0 \tag{31}$$

Viewing (30) and (31) as identities, valid for arbitrary values of δx and $d\nu$, it follows that the coefficients of δx and $d\nu$ must vanish; this yields differential equations for P , R , Q , h , and g . Also, if we require that (30) and (31) be equivalent to (15) and (16) at the terminal time, we obtain boundary conditions for P , R , Q , h , and g :

$$\dot{P} = -A^TP-PA-PBP+C \;;\; P(t_1) = [\phi_{xx}+\nu^T\psi_{xx}]_{t=t_1} \tag{32}$$

$$\dot{R} = -(A^T+PB)R \;;\; R(t_1) = [\psi_x^T]_{t=t_1} \tag{33}$$

$$\dot{Q} = -R^TBR \;;\; Q(t_1) = 0 \tag{34}$$

$$\dot{h} = -(A^T+PB)h-Pv+w \;;\; h(t_1) = 0 \tag{35}$$

$$\dot{g} = R^T(Bh+v) \;;\; g(t_1) = 0 \tag{36}$$

Note that (32) is a nonlinear matrix differential equation (a matrix Ricatti equation), while (33) is a linear matrix differential equation using the solution of (32); (34) is a matrix quadrature using the solution of (33); (35) is a linear vector differential equation using the solution of (32), and (36) is a vector quadrature using the solution of (35).

By integrating (32)-(36) backward along with (8) from t_1 to t_0 (a "backward sweep") we generate all possible solutions to (11)-(13) that satisfy the terminal conditions (15)-(16). We may think of (25)-(26) as "boundary conditions" at time $t < t_1$ that are <u>equivalent</u> to the boundary conditions (15)-(16) at time $t = t_1$. Thus the boundary conditions at the terminal time are "swept" backward to the initial time: a "forward sweep" then generates the required particular solution that also satisfies the initial conditions (14). This is precisely the approach taken by Bryson and Frazier (Ref. 6) to solve the linear smoothing problem except that the sweeps occur in the opposite order; the "forward sweep" is the Kalman-Bucy filter which involves a matrix Ricatti equation, and the "backward sweep" gives the smoothing solution that satisfies the terminal conditions.

After completing the backward sweep, the required values of $d\nu$ in terms of the desired infinitesimal changes $\delta H_u(t)$, $\delta x(t_0)$, and $\delta\psi$ can be obtained using (26):

$$d\nu = [Q^{-1}(\delta\psi-g-R^T\delta x)]_{t=t_0} \tag{37}$$

Having these values of $d\nu$, we could, in principle, substitute them into (29) and integrate these equations forward with (25) and (17) to find $\delta x(t)$, $\delta\lambda(t)$, and $\delta u(t)$ (a "forward sweep").

Alternatively, using (25) we could regard (17) as a linear feedback law for determining $\delta u(t)$:

$$\delta u(t) = -H_{uu}^{-1}(t)\{[H_{ux}(t)+f_u^T(t)P(t)]\delta x(t)$$
$$+ f_u^T(t)R(t)d\nu - \delta H_u(t) + f_u^T(t)g(t)\} \tag{38}$$

Note that $d\nu$ in (38) may be evaluated at the initial time $t = t_0$ as was done in (37) <u>or</u> we may evaluate it at several intermediate times in the manner of a sampled-data feedback law <u>or</u> we may evaluate it continuously in the manner of a continuous feedback law. If we do evaluate $d\nu$ continuously, then (38) becomes

$$\delta u(t) = -H_{uu}^{-1}\{[H_{ux}+f_u^T(P-RQ^{-1}R^T)]\delta x + [f_u^TRQ^{-1}]\delta\psi + [-\delta H_u+f_u^T(h-RQ^{-1}g)]\} \quad (38a)$$

Now, the first term in square brackets on the right hand side of (38a) is a linear feedback on deviations $\delta x(t)$ from the nominal state variable histories and will keep $\delta H_u(t) = 0$, $\delta\psi = 0$, for $\delta x(t_0) \neq 0$. The second term in square brackets is the forcing function necessary to produce the desired changes $\delta\psi$ while holding $\delta H_u(t) = 0$. The third term in square brackets is the forcing function necessary to produce the desired changes $\delta H_u(t)$ while holding $\delta\psi = 0$; it is a linear functional of $\delta H_u(t)$, and vanishes if $\delta H_u(t) = 0$. We could, therefore, integrate (2) forward (a "forward sweep"), using (38) in

$$u(t) = u_{old}(t) + \delta u(t) \quad (39)$$

$$\delta x(t) = x(t) - x_{old}(t) \quad (40)$$

The advantage of this procedure over previous gradient procedures is that we have separate, precise control over the desired changes $\delta H_u(t)$ and $\delta\psi$. By repeating this forward-backward sweep several times we can bring $H_u(t)$ and $\psi[x(t_1),t_1]$ precisely to zero while increasing the performance index; the required number of steps depends on the successful range of linearization of (11)-(16). We suggest that if N steps are to be used, it would be reasonable to choose

$$\delta H_u^{(r)}(t) = -\epsilon^{(r)}H_u^{(r-1)}(t) \quad (41)$$

$$\delta\psi^{(r)} = -\epsilon^{(r)}\psi^{(r-1)}[x(t_1),t_1] \quad (42)$$

where $\epsilon^{(r)} = r/N$ and r is the step number; in this way, larger and larger reductions in the "residuals" are taken each step and, on the last step, the whole remaining correction is made, bringing $H_u(t)$ and ψ precisely to zero.

LOCAL OPTIMALITY - GENERALIZED JACOBI TEST AND CONJUGATE POINTS

When we have succeeded in bringing $H_u(t) = 0$ and $\psi[x(t_1),t_1] = 0$, we have generated an admissible extremal path. For this case, the feedback law (38a) simplifies to:

$$\delta u(t) = -H_{uu}^{-1}(t)[H_{ux}+f_u^T(P-RQ^{-1}R^T)]\delta x(t) \quad (43)$$

since $\delta\psi = 0$ and $\delta H_u(t) = 0$ implies that $v = w = h = g = 0$ (see Eqns. (23), (24), (35), (36)). If the symmetric $m\times m$ matrix $H_{uu}(t)$ is positive (or negative) definite and the symmetric $n\times n$ matrix $P-RQ^{-1}R^T$ is finite over the semi-open interval $t_0 \leq t < t_1$, then (43) indicates $\delta u(t) = 0$ if $\delta x(t_0) = 0$ and we are assured that we have generated a path that is at least a local optimal path. This is a generalized Jacobi test; if $P-RQ^{-1}R^T$ becomes infinite at some point this is called a conjugate point to the terminal manifold $\psi[x(t_1),t_1] = 0$. An extremal path is not an optimal path if it contains a conjugate point (see e.g. Ref. 4).

INTERPRETATION OF THE MATRICES P, Q, AND R

Let us define a return function. $V(\nu,u,x,t)$ which is the value of $\bar{J}$ in (7) when starting from state x at time $t = t_1$ using the control functions $u(t)$ in (2) and the control parameters ν . Infinitesimal variations away from a given set of initial conditions, $\delta x(t)$, and infinitesimal changes in the control parameters, $d\nu$, while holding $\delta H_u(t) = 0$, will produce an infinitesimal change in the return function, δV , given by

$$\delta V = \begin{bmatrix}\lambda^T(t),\psi^T[x(t_1),t_1]\end{bmatrix}\begin{bmatrix}\delta x(t)\\ d\nu\end{bmatrix} + \frac{1}{2}[\delta x^T(t),d\nu^T]\begin{bmatrix}P(t),R(t)\\ R^T(t),Q(t)\end{bmatrix}\begin{bmatrix}\delta x(t)\\ d\nu\end{bmatrix} \quad (44)$$

From (44) it is clear that

$$\lambda^T(t) = \frac{\partial V}{\partial x(t)} \ , \ \psi^T = \frac{\partial V}{\partial \nu} \ ,$$

$$P(t) = \frac{\partial^2 V}{\partial x(t)\partial x(t)} \ , \ R = \frac{\partial^2 V}{\partial \nu \partial x(t)} \ , \ Q = \frac{\partial^2 V}{\partial \nu^2} \quad (45)$$

From (26), or (45)-(46), we can also write

$$R^T(t) = \frac{\partial \psi}{\partial x(t)} \ , \ Q(t) = \frac{\partial \psi}{\partial \nu} \quad (46)$$

and we note these quantities are similar to the steepest-ascent quantities $\lambda^{(\psi)}(t)$ and $I_{\psi\psi}(t)$ of Bryson and Denham (Ref. 7).

If the path is extremal ($H_u(t) = 0$) , and satisfies the terminal conditions ($\psi[x(t_1),t_1] = 0$), then $V = V(x,t)$ is the optimal return function of Hamilton-Jacobi-Bellman theory (see e.g. Ref. 8). Equation (44), using (26) with $\delta\psi = 0$, $g = 0$, to eliminate $d\nu$ becomes

$$\delta V = \lambda^T\delta x + \frac{1}{2}\,\delta x^T(P-RQ^{-1}R^T)\delta x \quad (47)$$

which gives the infinitesimal change in the optimal return function for infinitesimal changes in the initial conditions $\delta x(t)$ holding the final conditions constant ($\delta\psi = 0$) .

SUMMARY FOR CASE WHERE FINAL TIME IS GIVEN EXPLICITLY

(A) Estimate the control functions $u(t)$ and integrate $\dot{x} = f(x,u,t)$ forward with given values of $x(t_0)$. Record the constants $\psi[x(t_1),t_1]$, and the functions $u(t)$, $x(t)$.

(B) Estimate the control parameters ν and integrate / $\dot{\lambda} = -f_x^T\lambda$ backward with $\lambda(t_1) = [\phi_x+\nu^T\psi_x]_{t=t_1}$, using $u(t)$, $x(t)$ to evaluate $f_x[x(t),u(t),t]$. Calculate $H = \lambda^Tf$ and its derivatives H_u , H_{uu} ,

H_{ux} , H_{xx} as you go. H_{uu}^{-1} must also be calculated. While doing so, one can verify that H_{uu} is positive (or negative) definite. If H_{uu} does not satisfy the appropriate condition, better estimates of $u(t)$ and ν are required in (A).

(C) Simultaneously with (B), integrate Eqns. (32)-(36) for P , Q , R , h , and g backward, using the derivatives of H from (B) and $\delta H_u(t)$ from (41). Record the forcing functions

$$H_{uu}^{-1}(t)[-\delta H_u(t)+f_u^T(t)h(t)] = U(t)$$

and the feedback gains

$$H_{uu}^{-1}(t)[H_{ux}(t)+f_u^T(t)P(t)] = K(t)$$

$$H_{uu}^{-1}(t)f_u^T(t)R(t) = L(t)$$

(D) Determine and record the parameters $d\nu$ from (37), i.e.

$$d\nu = Q^{-1}(t_0)[\delta\psi-g(t_0)-R^T(t_0)\delta x(t_0)]$$

(E) Repeat (A) using the improved estimates of $u(t)$

$$u(t) = u_{old}(t) + U(t) - K(t)[x(t)-x_{old}(t)] - L(t)d\nu$$

(F) Repeat (B), (C), and (D) using the improved estimates of ν ,

$$\nu = \nu_{old} + d\nu$$

(G) Repeat (E) and (F) until $H_u(t) = 0$, $\psi[x(t_1),t_1] = 0$.

CASE WHERE FINAL TIME IS GIVEN IMPLICITLY

If the final time, t_1 , is given implicitly in (4), then it is necessary to estimate t_1 for the first forward sweep, in addition to $u(t)$ and ν . A few additional equations must be integrated on the backward sweep in order to determine the required dt_1 for the next forward sweep.

The development is the same as in the previous case through Eqn. (10); at that point an additional necessary condition is required to determine the final time, namely the transversality condition

$$\Omega[x(t_1),t_1] = [\phi_t+\phi_x f+L]_{t=t_1} = 0 \qquad (48)$$

The development is again the same up to Eqns. (15) and (16) which are replaced by the following:

$$\begin{bmatrix}\delta\lambda(t_1)\\ d\psi\\ d\Omega\end{bmatrix} = \begin{bmatrix}\phi_{xx}(t_1),\psi_x^T(t_1),m(t_1)\\ \psi_x(t_1)\ ,\ 0\ \ ,n(t_1)\\ m^T(t_1)\ ,n^T(t_1),\alpha(t_1)\end{bmatrix}\begin{bmatrix}\delta x(t_1)\\ d\nu\\ dt_1\end{bmatrix} + \begin{bmatrix}0\\ 0\\ H_u H_{uu}^{-1}\delta H_u\end{bmatrix} \qquad (49)(50)(51)$$

where

$$m^T(t_1) = [\Omega_x-H_u H_{uu}(H_{ux}+f_u^T\phi_{xx})]_{t=t_1} \qquad (52)$$

$$n^T(t_1) = [\frac{D\psi^T}{Dt} -H_u H_{uu}^{-1}f_u^T\psi_x^T]_{t=t_1} \qquad (53)$$

$$\alpha(t_1) = [\frac{D}{D}(\frac{D\phi}{Dt}+L)]_{t=t_1} \qquad (54)$$

and

$$\frac{D()}{Dt} = \frac{\partial()}{\partial t} + \frac{\partial()}{\partial x}\dot{x} + \frac{\partial()}{\partial u}\dot{u}$$

$$\dot{u} = -H_{uu}^{-1}(f_u^T\phi_{xt}+H_{xt}+f_u^T\phi_{xx}f+H_{ux}f)$$

Equations (17)-(24) are still applicable but, in view of (49)-(51), the inhomogeneous Ricatti trsns-formation beginning at (25) must be generalized to the following:

$$\begin{bmatrix}\delta\lambda(t)\\ d\psi\\ d\Omega\end{bmatrix} = \begin{bmatrix}P(t)\ ,R(t)\ ,m(t)\\ R^T(t),Q(t)\ ,n(t)\\ m^T(t),n^T(t),\alpha(t)\end{bmatrix}\begin{bmatrix}\delta x\\ d\nu\\ dt_1\end{bmatrix} + \begin{bmatrix}h(t)\\ g(t)\\ \beta(t)\end{bmatrix} \qquad (55)(56)(57)$$

Differentiating (55)-(57) with respect to time, using the fact that $d\psi$, $d\Omega$, $d\nu$, and dt_1 are constsnts, we obtain

$$\begin{bmatrix}\delta\dot{\lambda}\\ 0\\ 0\end{bmatrix} = \begin{bmatrix}\dot{P}\ ,\dot{R}\ ,\dot{m}\\ \dot{R}^T,\dot{Q}\ ,\dot{n}\\ \dot{m}^T,\dot{n}^T,\dot{\alpha}\end{bmatrix}\begin{bmatrix}\delta x\\ d\nu\\ dt_1\end{bmatrix} + \begin{bmatrix}P\\ R^T\\ m^T\end{bmatrix}\delta\dot{x} + \begin{bmatrix}\dot{h}\\ \dot{g}\\ \dot{\beta}\end{bmatrix} \qquad (58)(59)(60)$$

Using (55) in (18) gives

$$\delta\dot{x} = (A+BP)\delta x + BRd\nu + BMdt_1 + Bh + v \qquad (61)$$

Using (55) in (19), together with (61), we can eliminate $d\dot{\lambda}$ and $\delta\dot{x}$ from (58)-(60), and obtain three equations like (30) and (31) in δx , $d\nu$, and dt_1 . These three equations sre satisfied identically if we choose P , Q , R , h , and g to aatisfy (32)-(36) and m , n , α to satisfy

$$\dot{m} + (A^T+PB)m = 0 \qquad (62)$$

$$\dot{n} = -R^T Bm \qquad (63)$$

$$\dot{\alpha} = -m^T Bm \qquad (64)$$

$$\dot{\beta} = -m^T(Bh+v) \qquad (65)$$

where the boundary conditions for m , n , α are given by (52)-(54). Note (62) is the same linear vector differential equation as (33) whereas (63) and (64) are simply quadratures.

If (62)-(65) are included in the backward integration sweep, then it is possible to solve for both $d\nu$ and dt_1 at $t = t_0$, using (56) and (57) where desired values of $d\psi$ and $d\Omega$ for the next step are introduced. The desired value of $\delta H_u(t)$ must be used in solving for h , g , and β from (35), (36), and (64.

REFERENCES

1. Breakwell, J. V., Speyer, J. L., and Bryson, A. E., Optimization and Control of Nonlinear Systems Using the Second Variation, S.I.A.M. Journal of Control, Ser. A., Vol. 1, No. 2, pp. 193-223, 1963.

2. Merriam, C. W., An Algorithm for the Iterative Solution of a Class of Two Point Boundary Value Problems, S.I.A.M. Journal of Control, Ser. A., Vol. 2, No. 1, pp. 1-10, 1964

3. Kelley, H. J., Kopp, R., and Moyer, G., A Trajectory Optimization Technique Based Upon the Theory of the Second Variation, AlAA Astrodynamics Conference, Yale University, New Haven, Connecticut, August 19-21, 1963.

4. Gelfand, I. M. and Fomin, S. V., Calculus of Variations, Chapter 6, Prentice Hall, 1963.

5. Rybicki, G. B. and Usher, P. D., The Generalized Ricatti Transformation as a Simple Alternative to lnvariant lmbedding, Smithsonian Astrophysical Laboratory, Cambridge, Mass., December 1964.

6. Bryson, A. E. and Frazier, M., Smoothing for Linear and Nonlinear Dynamic Systems, Proceedings Optimum System Synthesis Conf., Wright-Patterson AFB, September 1962, USAF Rpt. ASD-TDR-63-119, 1963.

7. Bryson, A. E. and Denham, W. F., A Steepest-Ascent Method for Solving Optimum Programming Problems, J. Appl. Mech., 29, 2, pp. 247-257, 1962.

8. Bellman, R. E. and Dreyfus, S. E., Applied Dynamic Programming, Princeton University Press, 1962.

APPENDlX

THE NEWTON-RAPHSON METHOD AND ITS APPLICATIONS TO ORDlNARY CALCULUS PROBLEMS

ln this Appendix the Newton-Raphson method is briefly stated. lt will be seen that the Newton-Raphson method applied to optimization problems becomes a second-order iterative scheme which can be applied in the neighborhood of a non-singular optimum in order to obtain rapid convergence.

The formulation of second order steepest-ascent methods may be based upon a simple extension of the Newton-Raphson method used to solve s set of simultaneous nonlinear equations. Suppose one wishes to find an n-vector $x = (x_1,..,x_n)$ such that

$$P(x) = 0 \qquad P = (P_1,...P_n) \tag{A1}$$

The Newton-Raphson method generates a sequence $(x^{(0)},x^{(1)},...)$ by means of an iterative relation (A2) .

$$x^{k+1} = x^k - [(\frac{\partial P}{\partial x})^{-1} P]_{x=x^k} \tag{A2}$$

The rationale for this is obtained by expanding $P(x)$ in a power series around x^k .

$$P(x^k+dx) = P(x^k) + \frac{\partial P}{\partial x}\Big|_{x=x_k} dx + O(dx^2) \tag{A3}$$

Setting $P(x^k+dx) = 0$, one sees that

$$dx = (\frac{\partial P}{\partial x})^{-1} P(x^k) + O(dx^2) \tag{A4}$$

By ignoring second and higher order terms on the right hand side of (A4) one obtains an estimate of the error in x within first order accuracy. Thus (A2) approximates the solution within a second order error. The method naturally assumes $\frac{\partial P}{\partial x}$ to be nonsingular in the region containing (x^k) and the solution.

The Newton-Raphson method may be extended to finding a local maximum of a function of several variables $f(x)$. If f is continuously differentiable, a local maximum x is characterized by being a solution to the following equations

$$f_{x_i} = 0 \qquad i = 1,...,n \tag{A5}$$

Applying the Newton-Raphson method to these equations, one arrives at a second-order steepest-ascent method by merely identifying f_{x_i} with P_i in (A2).

The method may be readily extended to problems with constraints. Suppose the maximum of f is wanted subject to the added constraint

$$g(x) = 0 \tag{A6}$$

ln place of this problem one may substitute the problem of extremizing $f+\lambda g$ with respect to x and λ as independent variables. This problem has no constraints and may be handled as the first case. An extremal is characterized by (A6) and

$$f_x + \lambda g_x = 0 \tag{A7}$$

Expanding (A6) sround a nominal solution (x^k,λ^k) one obtains the following set of linear, inhomogeneous equations to solve:

$$\begin{aligned} 0 &= (f_x+\lambda g_x)\Big|_{x^k,\lambda^k} + (f_{xx}+\lambda g_{xx})dx + g_x d\lambda \\ 0 &= g\Big|_{x^k} + g_x\, dx \end{aligned} \tag{A8}$$

Solving (A8) yields corrections dx and $d\lambda$, and the second order steepest-ascent method becomes

$$\begin{aligned} x^{k+1} &= x^k + dx \\ x^{k+1} &= \lambda^k + d\lambda \end{aligned} \tag{A9}$$

Several cautions must be exercised. One is that dx must be small in order to guarantee convergence, which implies that the original error should not be too big. Secondly, the nominal and the maximum must be non-singular and normal. This is necessary to guarantee the inversion of the basic equations. The non-singularity condition guarantees that one can solve for dx . The normality condition guarantees that one can solve for $d\lambda$. Thirdly, one should note that the second-order steepest-ascent method seeks out stationary solutions, regardless of whether they are local minima, local maxima, or saddle points. ln order to be sure that the sequence converges to the desired extremum, the eigenvalues of the second derivative matrix must be checked. This can be seen for the problem without constraints by substituting (A2) with $P = f_x$ into a power series for f around x^k .

$$f(x^{k+1}) = f(x^k) - \frac{1}{2} f_x f_{xx}^{-1} f_x^1 + O(f_x) \tag{A10}$$

ln order to guarantee that $f(x^{k+1}) > f(x^k)$, it is necessary to assume $f_{xx} < 0$.

LIST K

ASD (ASRNRE-3, Mr. Paul Springer)
Wright-Patterson AFB, Ohio

Director
U.S. Army Ballistic Research Labs.
Aberdeen Proving Ground, Maryland
Attn: Ballistic Measurements Lab.

Airborne Instruments Laboratory, Inc.
Division of Cutler Hammer
Walt Whitman Road
Melville, L.I., New York
Attn: Dr. E. Fubini, Director
Research and Engineering Div.

Stanford Research Institute [2]
Menlo Park, California
Attn: Dr. D.R. Schenck, Assistant Dir.
Division of Engineering Research

Bell Aircraft Corporation
P.O. Box 1
Buffalo 5, New York
Attn: Eunice P. Haselton, Librarian

Goodyear Aircraft Corporation
1210 Massillon Road
Akron 15, Ohio
Attn: Library, Plant C

Lockheed Aircraft Corporation
California Division
P.O. Box 551
Burbank, California
Attn: Mr. H. Rempt

Convair, A Division of General Dynamics Corporation
3165 Pacific Highway
San Diego 12, California
Attn: Mr. R.E. Hemer, Head
Electronics Research

Ryan Aeronautical Company
2701 Harbor Drive
Lindbergh Field
San Diego 12, California
Attn: Mr. J. R. Giantvalley

Boeing Airplane Company
Aero Space Division
P.O. Box 3707
Seattle 24, Washington
Attn: R.R. Barber, Library
Supervisor

General Electric Company
Building 3 - Room 143-1
Electronics Park
Syracuse, New York
Attn: Yolanda Burke
Documents Library

Chu Associates
P.O. Box 387
Whitcomb Avenue
Littleton, Massachusetts

AFCRL, OAR
L.G. Hanscom Field
Bedford, Massachusetts
Attn: Carlyle J. Sletten, CRD [3]
Attn: Contract Files, CRD [2]
Attn: R. Shore, CRD [5]

Chief of Naval Operations
Department of the Navy
Attn: Op-413-B21
Washington 25, D.C.

Commander
U.S. Naval Air Missile Test Center
Point Mugu, California
Attn: Code 366

Office of Naval Research (Code 427)
Department of the Navy
Washington 25, D.C.

California Institute of Technology
Jet Propulsion Laboratory
Pasadena 4, California
via: District Chief, Los Angeles Ordnance District, 55S. Grand Ave., Pasadena California-Attn: ORDEV-M

ESD (ESRDW, Major John J. Hobsen)
L.G. Hanscom Field, Bedford, Mass.

Cornell University
School of Electrical Engineering
Ithaca, New York
Attn: Prof. C.C. Dalman

Georgia Technical Research Institute
Engineering Experiment Station
722 Cherry Street, N.W.
Atlanta, Georgia
Attn: Dr. James E. Boyd

The Johns Hopkins University [2]
Radiation Laboratory
1315 St. Paul Street
Baltimore, Maryland

The Johns Hopkins University [2]
Applied Physics Laboratory
8621 Georgia Avenue
Silver Springs, Maryland

McGill University
Montreal, Canada
Attn: Prof. G.A. Woonton, Director
The Eaton Electronics Res. Lab.

The University of Michigan [2]
Engineering Research Institute
Willow Run Laboratories
Willow Run Airport
Ypsilanti, Michigan

New York University [2]
Institute of Mathematical Sciences
25 Waverly Place, Room 802
New York 3, New York
Attn: Prof. Morris Kline

The Pennsylvania State University
Department of Electrical Engineering
University Park, Pennsylvania

The University of Texas
Defense Research Laboratory
Austin, Texas
Attn: Claude W. Horton, Physics Library

Polytechnic Institute of Brooklyn [2]
Microwave Research Institute
55 Johnson Street, Brooklyn, New York
Attn: Dr. Arthur A. Oliner

Case Institute of Technology
Cleveland, Ohio
Attn: Prof. S. Seeley

Purdue University
Department of Electrical Engineering
Lafayette, Indiana
Attn: Dr. Schultz

University of Tennessee
Ferris Hall
W. Cumberland Avenue
Knoxville 16, Tennessee

Dr. U. Ilne
University of Kansas
Electrical Engineering Department
Lawrence, Kansas

Ohio State University Research Foundation
Antenna Laboratory
Attn: Dr. C. N. Walter
1314 Kinnear Road
Columbus 12, Ohio

LIST C-D

Los Angeles Air Force Station
Space Systems Division
Los Angeles, California 90045
Air Force Unit Post Office
Attn: SSSD

Library of Congress
Aerospace Information Division
Washington 25, D.C.

Library of Congress
Exchange and Gift Division
Washington 25, D.C.

NASA
Goddard Institute for Space Studies (Library)
475 Riverside Drive
New York 27, New York

NBS (Library)
Boulder, Colorado

Smithsonian Astrophysical Observatory (Library)
60 Garden Street
Cambridge, Massachusetts

AFCRL (CRMXR)
L.G. Hanscom Field
Bedford, Massachusetts

Dr. N. L. Krisberg
Boeing Aerospace Division
Box 3707
Seattle 24, Washington

Dr. O. G. Heywood, Jr.
Huyck Corporation
777 Summer Street
Stamford, Connecticut

Dr. L.F. Carter
SDC
2500 Colorado Avenue
Santa Monica, California

Office of Naval Research
Branch Office (London)
Navy 100, Box 39
F.P.O. New York, New York

Dr. Carl Kaplan
The Johns Hopkins University
2500 West Rogers Avenue
Baltimore 15, Maryland

Prof. Henry C. Houghton
Department of Meteorology
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139

Prof. C.S. Pittendrigh
Department of Biology
Princeton University
Princeton, New Jersey

Prof. Leonard Schiff
Department of Physics
Stanford University
Stanford, California

Dr. Joseph Kaplan
Department of Physics
U.C.L.A.
Los Angeles 24, California

Prof. Frederick Seitz
Department of Physics
University of Illinois
Urbana, Illinois

AIAA
Technical Information Service (Library)
750 Third Avenue
New York, New York

NCAR (Library)
Boulder, Colorado

Dr. Mark Kac
Rockefeller Institute
New York 21, New York

Scientific Information Officer [3]
Defence Research Staff
3100 Massachusetts Avenue, N.W.
Washington 8, D.C.

Defence Research Member [3]
Canadian Joint Staff
2450 Massachusetts Avenue, N.W.
Washington 8, D.C.

National Research Council (Library)
Ottawa 2, Ontario, Canada

Dr. David B. Langmuir
STL
One Space Park
Redondo Beach, California

AFSC-STLO (RTSAS)
452 Deguigne
Sunnyvale, California 94086

USAF Academy (Library)
Colorado Springs, Colorado

Air University (Library)
Maxwell AFB, Alabama

USAF (AFGCS-SA)
Washington 25, D.C.

USAF (AFRST)
Washington 25, D.C.

OAR (RRC)
Washington 25, D.C.

OAR (RRO)
Washington 25, D.C.

OAR (RRY)
Washington 25, D.C.

AFSWC (Tech. Library)
Kirtland AFB, New Mexico

FUSRL
USAF Academy
Colorado Springs, Colorado

OAR (RRRT)
Holloman AFB, New Mexico

ADC (Operations Analysis Office)
Ent AFB, Colorado

SAC (Operations Analysis Office)
Offutt AFB, Nebraska

TAC (Operations Analysis Office)
Langley AFB, Virginia

BSD (Tech. Library)
Norton AFB, California

AEDC (Tech. Library)
Arnold AFB, Tennessee

AFFTC (Tech. Library)
Edwards AFB, California

AFMTC (AFMTC Tech. Library-MU-135)
Patrick AFB, Florida
Unclassified Material
AFMTC (MTBAT)
Patrick AFB, Florida
Classified Material

APGC (Library)
Eglin AFB, Florida

Systems Engineering Group
Deputy for Systems Engineering
Directorate of Tech. Pub. and SPEC (SEPRR)
Wright-Patterson AFB, Ohio 45433

AFSC-STLO (RSTAC)
O'Hare International Aleport
Box 8758
Chicago, Illinois 60666

RTD (Scientific Director)
Bolling AFB, D.C.

RAOC (RAALD, Library)
Griffiss AFB, New York

USAFSS (ORD)
San Antonio, Texas 78241

AFSC-STLO (RTSUM)
68 Albany Street
Cambridge, Massachusetts 02139

AFSC-STLO (RSTAL)
AF Unit Post Office
Los Angeles, California 90045

AWS (AWSSS/TIPD)
Scott AFB, Illinois

AFIT (MCLI, Library)
Building 125, Area B
Wright-Patterson AFB, Ohio

RTD (APX)
Wright-Patterson AFB, Ohio

RTD (AWX)
Wright-Patterson AFB, Ohio

RTD (FDX)
Wright-Patterson AFB, Ohio

RTD (MAX)
Wright-Patterson AFB, Ohio

AFSC STLO (RTSAB)
Building 128 South
Waltham Federal Center
Waltham, Massachusetts 07130

Office of the Chief Signal Officer (SIGRD-4A-2)
Washington 25, D.C.

Army Missile Command
Redstone Scientific Information Center
Redstone Arsenal, Alabama

Army Research Office (Library)
Arlington Hall Station, Virginia

U.S. Military Academy (Library)
West Point, New York

Army Electronic Proving Ground (Tech. Library)
Fort Huachuca, Arizona

USASRDL (Tech. Doc. Center)
Ft. Monmouth, New Jersey

Evans Signal Laboratories
Technical Documents Center
Belmar, New Jersey

ODDR+E (Library)
Room 3C-128
The Pentagon
Washington 25, D.C.

ARPA (Library)
The Pentagon
Washington 25, D.C.

DCA (Library)
Washington 25, D.C.

DIA (Library)
Washington 25, D.C.

Defence Documentation Center [20]
Cameron Station
Alexandria, Virginia 22314

DASA (Library)
Washington 25, D.C.

U.S. Atomic Energy Commission
Headquarters Library
Reports Section
Room G-017
Washington, D.C. 20545

Los Alamos Scientific Laboratories
Los Alamos, New Mexico

FAA
Bureau of Research and Development
300 Independence Ave. S.W.
Washington, D.C. 20553

Government Printing Office
Library (Div. of Public Documents)
Washington 25, D.C.

Scientific and Technical Information Facility
Attn: NASA Representative
P.O. Box 5700
Bethesda, Maryland 20014

Scientific and Technical Information Facility
NASA Representative (S-AK/DL)
Box 5700
Bethesda, Maryland

NASA/AMES Research Center
Technical Library
Moffett Field, California

NASA/Flight Research Center
Technical Library
Box 273
Edwards, California

NASA/Goddard Space Flight Center
Technical Library
Greenbelt, Maryland

NASA/UPL (Library)
4800 Oak Grove Drive
Pasadena 3, California

NASA/Langley Research Center
Technical Library
Langley Station
Hampton, Virginia

NASA/Lewis Research Center
Technical Library
21000 Brookpark Road
Cleveland 35, Ohio

NASA/Manned Spacecraft Center
Technical Library
Houston 1, Texas

NASA/Marshall Space Flight Center
Technical Library
Huntsville, Alabama

National Science Foundation
1951 Constitution Ave., N.W.
Washington, D.C. 20235

U.S. Weather Bureau (Library)
Washington 25, D.C.

CIA
Office of Scientific Intelligence (Doc. Div)
2430 E Street, N.W.
Washington 25, D.C.

Chief of Naval Operations (OP-413-B21)
Washington 25, D.C.

Naval Postgraduate School (Tech. Library)
Monterey, California

Bureau of Naval Weapons (DLI-31, Library)
Washington 25, D.C.

Bureau of Ships (Code 320)
Washington 25, D.C.

ONR (Library)
Washington 25, D.C.

NRL (Code 2027, Library)
Washington 25, D.C.

Naval Air Development Center (Library)
Johnsville, Pennsylvania

Naval Missile Center (Library)
Point Mugu, California

Navy Electronics Laboratory (Library)
San Diego 52, California

Naval Ordnance Laboratory (Tech. Library)
White Oak, Silver Springs 19, Maryland

NOTS (Library)
China Lake, California

Aerospace (Library)
2400 E. El Segundo Boulevard
El Segundo, California

Battelle Memorial Institute (Library)
505 King Avenue
Columbus, Ohio

MITRE (Library)
Middlesex Turnpike
Bedford, Massachusetts

U.S. Naval Academy (Library)
Annapolis, Maryland

AFCRL (CRC)
L.G. Henscom Field
Bedford, Massachusetts

AFCRL (CRT)
L.G. Henscom Field
Bedford, Massachusetts

AFCRL (CRTE)
L.G. Henscom Field
Bedford, Massachusetts

AFCRL (CRTPM)
L.G. Henscom Field
Bedford, Massachusetts

AFCRL (CRR)
L.G. Henscom Field
Bedford, Massachusetts

AFCRL (CRMXLR) [3]
L.G. Henscom Field
Bedford, Massachusetts

AFCRL (CRMXRS) [10]
L.G. Henscom Field
Bedford, Massachusetts

ITT Research Institute (Library)
10 W 35 Street
Chicago, Illinois

NAS/NRC (Library)
2101 Constitution Avenue, N.W.
Washington 25, D.C.

RAND (Library)
1700 Main Street
Santa Monica, California

Unclassified
Security Classification

DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)*
Cruft Laboratory
Division of Engineering and Applied Physics
Harvard University, Cambridge, Massachusetts

2a. REPORT SECURITY CLASSIFICATION
Unclassified

2b. GROUP

3. REPORT TITLE

A SUCCESSIVE SWEEP METHOD FOR SOLVING OPTIMAL PROGRAMMING PROBLEMS

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*
Interim technical report

5. AUTHOR(S) *(Last name, first name, initial)*
Mc Reynolds, Stephen R. and Bryson, Arthur E.

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
2 March 1965	5	8

8a. CONTRACT OR GRANT NO.	9a. ORIGINATOR'S REPORT NUMBER(S)
Nonr-1866(16) b. PROJECT NO. NR-372-012 c. TASK d.	Cruft Laboratory Technical Report No. 463 9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)*

10. AVAILABILITY/LIMITATION NOTICES
Qualified requesters may obtain copies of this report from DDC.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
Accepted by Joint Automatic Control Conference, to be held at Rensselaer Polytechnic Inst., Troy, N.Y., June 1965	Joint Services Electronics Program under Office of Naval Research Contract Nonr-1866(16)

13. ABSTRACT

An automatic, finite-step numerical procedure is described for finding exact solutions to nonlinear optimal programming problems. The procedure represents a unification and extension of the steepest-descent, and second variation techniques.

The procedure requires the backward integration of the usual adjoint-vector differential equations plus certain matrix differential equations. These integrations correspond, in the ordinary calculus, to finding the first and second derivatives of the performance index respectively. The matrix equations arise from an inhomogeneous Ricatti transformation, which generates a linear "feedback control law" that preserves the gradient histories, $H_u(t)$, on the next step or permits changing them by controlled amounts, while also changing terminal conditions by controlled amounts. Thus, in a finite number of steps, the gradient histories can be made identically zero, as required for optimality, and the terminal conditions satisfied exactly. One forward plus one backward sweep, correspond to one step in the Newton-Raphson technique for finding maxima and minima in the ordinary calculus.

DD FORM 1 JAN 64 1473

Unclassified
Security Classification

Unclassified

Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Optimization Calculus of variations Dynamic programming Optimal programming Numerical analysis Ricatti transformation Newton-Raphson technique Feedback control						

INSTRUCTIONS

1. ORIGINATING ACTIVITY: Enter the name and address of the contractor, subcontractor, grantee, Department of Defense activity or other organization *(corporate author)* issuing the report.

2a. REPORT SECURITY CLASSIFICATION: Enter the overall security classification of the report. Indicate whether "Restricted Data" is included. Marking is to be in accordance with appropriate security regulations.

2b. GROUP: Automatic downgrading is specified in DoD Directive 5200.10 and Armed Forces Industrial Manual. Enter the group number. Also, when applicable, show that optional markings have been used for Group 3 and Group 4 as authorized.

3. REPORT TITLE: Enter the complete report title in all capital letters. Titles in all cases should be unclassified. If a meaningful title cannot be selected without classification, show title classification in all capitals in parenthesis immediately following the title.

4. DESCRIPTIVE NOTES: If appropriate, enter the type of report, e.g., interim, progress, summary, annual, or final. Give the inclusive dates when a specific reporting period is covered.

5. AUTHOR(S): Enter the name(s) of author(s) as shown on or in the report. Enter last name, first name, middle initial. If military, show rank and branch of service. The name of the principal author is an absolute minimum requirement.

6. REPORT DATE: Enter the date of the report as day, month, year, or month, year. If more than one date appears on the report, use date of publication.

7a. TOTAL NUMBER OF PAGES: The total page count should follow normal pagination procedures, i.e., enter the number of pages containing information.

7b. NUMBER OF REFERENCES: Enter the total number of references cited in the report.

8a. CONTRACT OR GRANT NUMBER: If appropriate, enter the applicable number of the contract or grant under which the report was written.

8b, 8c, & 8d. PROJECT NUMBER: Enter the appropriate military department identification, such as project number, subproject number, system numbers, task number, etc.

9a. ORIGINATOR'S REPORT NUMBER(S): Enter the official report number by which the document will be identified and controlled by the originating activity. This number must be unique to this report.

9b. OTHER REPORT NUMBER(S): If the report has been assigned any other report numbers *(either by the originator or by the sponsor)*, also enter this number(s).

10. AVAILABILITY/LIMITATION NOTICES: Enter uny limitations on further dissemination of the report, other than those imposed by security classification, using standard statements such as:

(1) "Qualified requesters may obtain copies of this report from DDC."

(2) "Foreign announcement and dissemination of this report by DDC is not authorized."

(3) "U. S. Government agencies may obtain copies of this report directly from DDC. Other qualified DDC users shall request through

_____."

(4) "U. S. military agencies may obtain copies of this report directly from DDC. Other qualified users shall request through

_____."

(5) "All distribution of this report is controlled. Qualified DDC users shall request through

_____."

If the report has been furnished to the Office of Technical Services, Department of Commerce, for sale to the public, indicate this fact and enter the price, if known.

11. SUPPLEMENTARY NOTES: Use for additional explanatory notes.

12. SPONSORING MILITARY ACTIVITY: Enter the name of the departmental project office or laboratory sponsoring *(paying for)* the research and development. Include address.

13. ABSTRACT: Enter an abstract giving a brief and factual summary of the document indicative of the report, even though it may also appear elsewhere in the body of the technical report. If additional space is required, a continuation sheet shall be attached.

It is highly desirable that the abstract of classified reports be unclassified. Each paragraph of the abstract shall end with an indication of the military security classification of the information in the paragraph, represented as *(TS)*, *(S)*, *(C)*, or *(U)*.

There is no limitation on the length of the abstract. However, the suggested length is from 150 to 225 words.

14. KEY WORDS: Key words are technically meaningful terms or short phrases that characterize a report and may be used as index entries for cataloging the report. Key words must be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location, may be used as key words but will be followed by an indication of technical context. The assignment of links, roles, and weights is optional.

Unclassified
Security Classification

DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)*	2a. REPORT SECURITY CLASSIFICATION 2b. GROUP
3. REPORT TITLE	
4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*	
5. AUTHOR(S) *(Last name, first name, initial)*	

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
8a. CONTRACT OR GRANT NO. b. PROJECT NO. c. TASK d.	9a. ORIGINATOR'S REPORT NUMBER(S) 9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)*	
10. AVAILABILITY/LIMITATION NOTICES		
11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY	

13. ABSTRACT

(continued)

As by-products, the procedure produces: (a) the functions needed to show that the program is, or is not, a local maximum, and (b) the feedback gain programs for neighboring optimal paths to the same, or a slightly different, set of terminal conditions. (Authors)

DD FORM 1 JAN 64 1473

Unclassified
Security Classification